MATH 128 Exam # 2B Solutions
Given on Oct. 19, 2009

Show all nontrivial calculations.

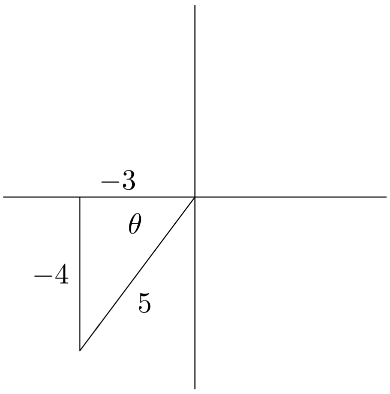
1. If $\cos\theta = -\frac{3}{5}$ and $\tan\theta > 0$, find the values of all trigonometric functions.

$\sin\theta = -\frac{4}{5}$ $\csc\theta = -\frac{5}{4}$

$\cos\theta = -\frac{3}{5}$ $\sec\theta = -\frac{5}{3}$

$\tan\theta = \frac{4}{3}$ $\cot\theta = \frac{3}{4}$

Since $\cos\theta$ is negative, θ is in either quadrant II or quadrant III. Since $\tan\theta$ is positive, it must be quadrant III. This gives us the picture below.



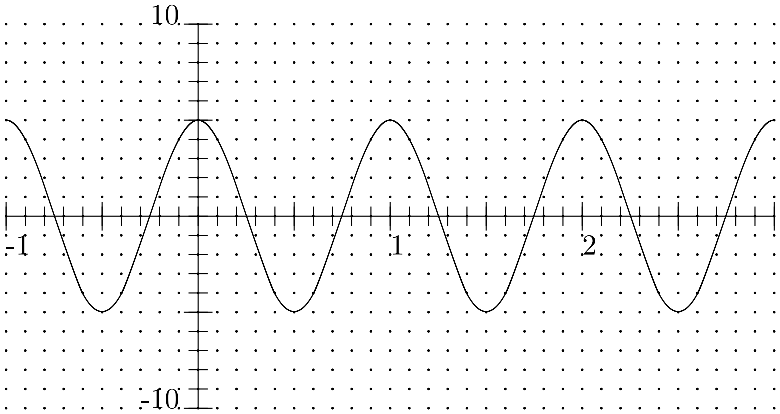
The base of the triangle is obtained from the Pythagorean theorem.

$$\begin{aligned} 5^2 &= (-3)^2 + y^2 \\ y^2 &= 5^2 - (-3)^2 \\ y &= \pm\sqrt{25-9} \\ y &= \pm 4 \end{aligned}$$

Since the angle is in quadrant III, $y = -4$.

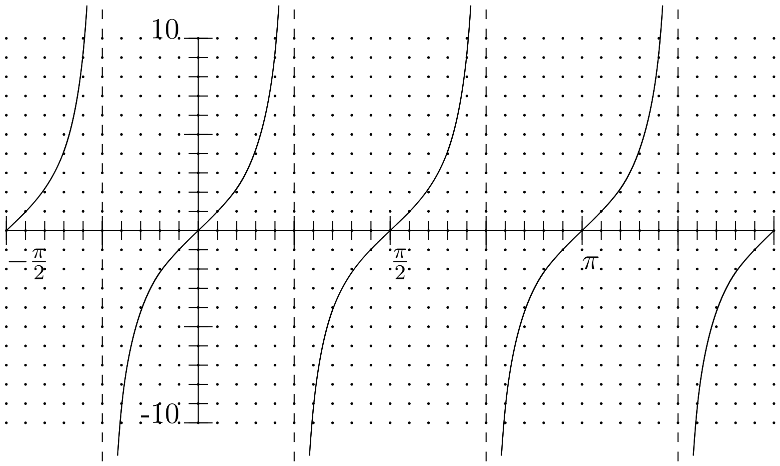
2. Graph at least one period of the following functions. *Number the axes.* Also find the amplitude (where applicable), the period and the phase shift.

(a) $f(x) = 5\cos(2\pi x)$



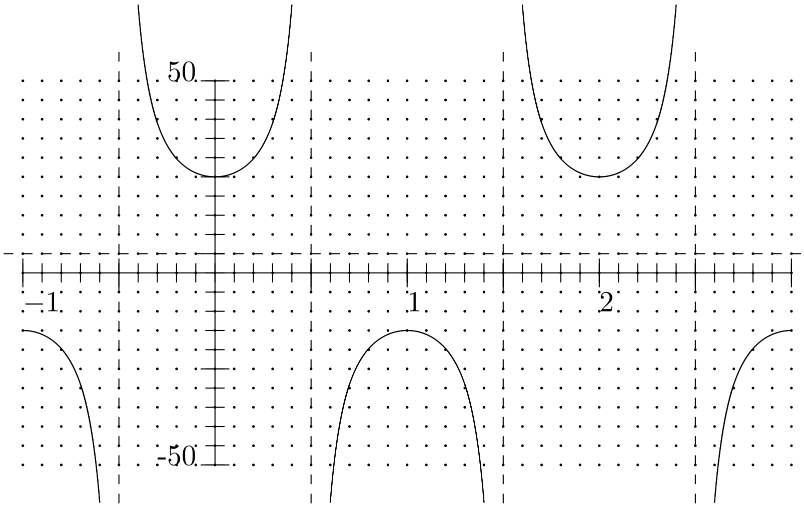
Amplitude $|5| = 5$ Period $\frac{2\pi}{2\pi} = 1$ Phase Shift none

(b) $g(x) = g(x) = 3\tan(2x)$



Amplitude none Period $\frac{\pi}{2}$ Phase Shift none

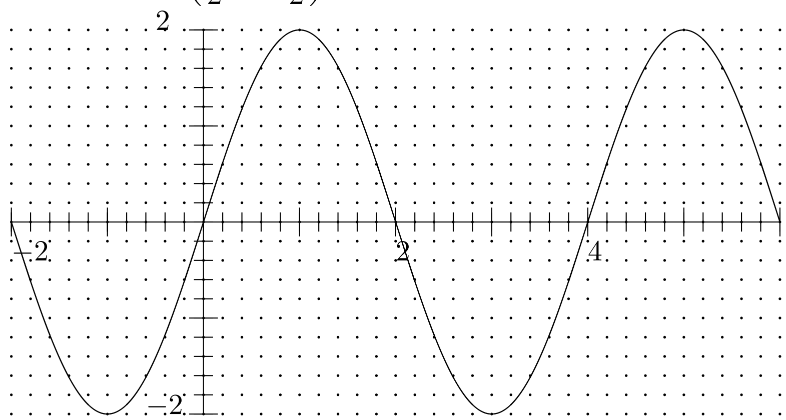
(c) $h(x) = 20\sec(\pi x) + 5$



Amplitude <u>none</u> Period $\frac{2\pi}{\pi} = 2$ Phase Shift <u>none</u>

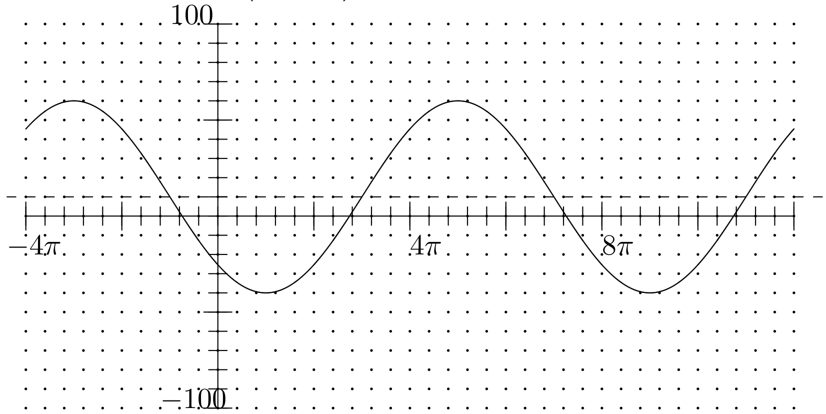
There is a vertical shift of +5.

(d) $f(x) = 2\cos\left(\frac{\pi}{2}x - \frac{\pi}{2}\right)$



Amplitude $|2| = 2$ Period $\frac{2\pi}{\pi/2} = 4$ Phase Shift $\frac{\pi/2}{\pi/2} = 1$

(e) $g(x) = -50\cos\left(\frac{x}{4} - \frac{\pi}{4}\right) + 10$

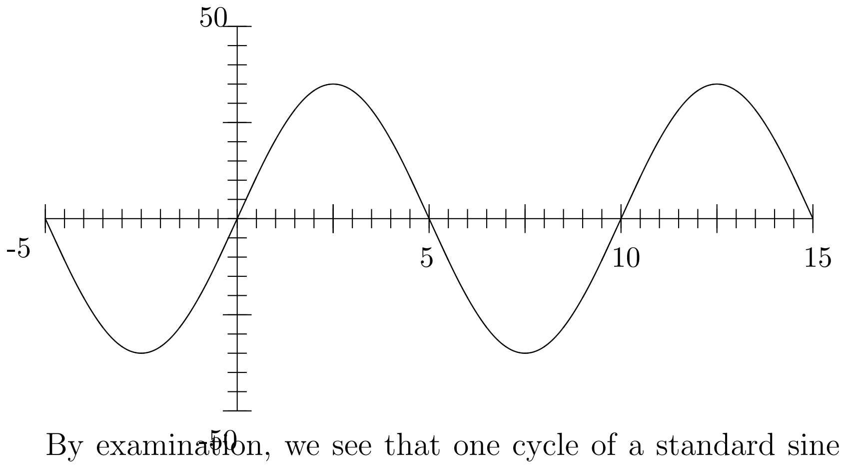


Amplitude $|-50| = 50$ Period $\frac{2\pi}{1/4} = 8\pi$ Phase Shift $\frac{\pi/4}{1/4} = \pi$

There is a vertical shift of $+10$

3. Find equations for the functions graphed.

(a)



By examination, we see that one cycle of a standard sine function begins at zero on the graph. So we can use a sine function with no phase shift. The form of the equation is $y = A\sin\omega x + B$. the maximum y-value is 35 and the minimum y-value is -35, so

$$A = \frac{\max - \min}{2} = \frac{35 - (-35)}{2} = 35 \qquad B = \frac{\max + \min}{2} = \frac{35 + (-35)}{2} = 0$$

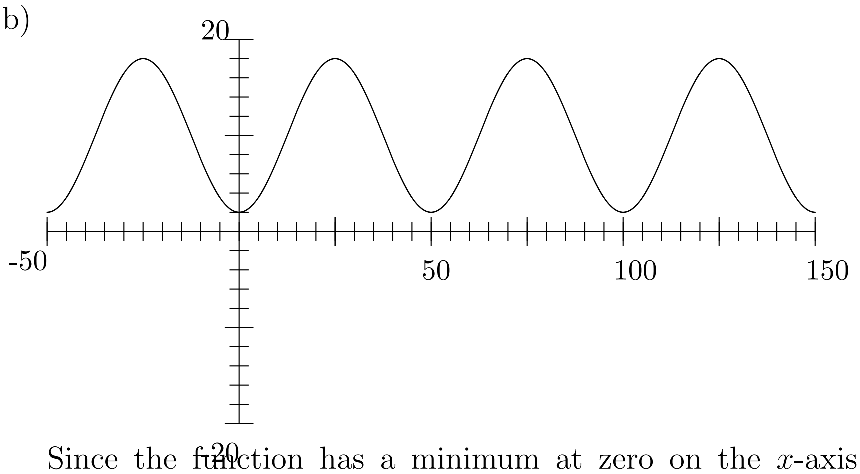
to determine ω, we note that one complete cycle starts at zero and ends at 10, so the period is $T = 10 - 0 = 10$. Then

$$\begin{aligned} T &= \frac{2\pi}{\omega} \\ \omega T &= 2\pi \\ \omega &= \frac{2\pi}{T} \\ \omega &= \frac{2\pi}{10} \end{aligned}$$

$$\omega = \frac{\pi}{5}$$

Then one equation of the function is $y = 35\sin\left(\frac{\pi}{5}x\right)$. Note that this is not the only way to represent this function. The same function could be rendered as $y = 35\cos\left(\frac{\pi}{5}x - \frac{\pi}{2}\right)$, for example.

(b)



Since the function has a minimum at zero on the x-axis, we can represent it as $y = -A\cos(\omega x) + B$ with no phase shift. The maximum y-value is 18 and the minimum y-value is 2.

$$A = \frac{\max - \min}{2} = \frac{18 - (2)}{2} = 8 \qquad B = \frac{\max + \min}{2} = \frac{18 + (2)}{2} = 10$$

One complete cycle occurs between 0 and 50 on the x-axis, so the period is $T = 50 - 0 = 50$.

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{50} = \frac{\pi}{25}$$

so one equation of the function is $y = -8\cos\left(\frac{\pi}{25}x\right) + 10$. Again, this is not unique. Another way of expressing the same function is $y = 8\cos\left(\frac{\pi}{25}x - \pi\right) + 10$.

4. Show that the functions $f(x) = \dfrac{2x+1}{x-3}$ and $g(x) = \dfrac{3x+1}{x-2}$ are inverse functions.

   Graphing the functions shows that they are one-to-one functions, since they satisfy both horizontal and vertical line tests.

$$\begin{aligned} f(g(x)) &= f\left(\frac{3x+1}{x-2}\right) \\ &= \frac{2\left(\frac{3x+1}{x-2}\right) + 1}{\left(\frac{3x+1}{x-2}\right) - 3} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\frac{6x+2}{x-2}+1}{\frac{3x+1}{x-2}-3} \\
&= \frac{\frac{6x+2}{x-2}+1}{\frac{3x+1}{x-2}-3} \times \frac{x-2}{x-2} \\
&= \frac{6x+2+(x-2)}{3x+1-3(x-2)} \\
&= \frac{6x+2+x-2}{3x+1-3x+6} \\
&= \frac{7x}{7} \\
&= x
\end{aligned}
$$

$$
\begin{aligned}
g\left(f(x)\right) &= g\left(\frac{2x+1}{x-3}\right) \\
&= \frac{3\left(\frac{2x+1}{x-3}\right)+1}{\left(\frac{2x+1}{x-3}\right)-2} \\
&= \frac{\frac{6x+3}{x-3}+1}{\frac{2x+1}{x-3}-2} \\
&= \frac{\frac{6x+3}{x-3}+1}{\frac{2x+1}{x-3}-2} \times \frac{x-3}{x-3} \\
&= \frac{6x+3+(x-3)}{2x+1-2(x-3)} \\
&= \frac{6x+3+x-3}{2x+1-2x+6} \\
&= \frac{7x}{7} \\
&= x
\end{aligned}
$$

5. Find inverse functions for the following.

   (a) $f(x) = 2x - 7$

$$
y = 2x - 7
$$

$$\begin{aligned} y+7 &= 2x \\ \frac{2x}{2} &= \frac{y+7}{2} \\ x &= \frac{y+7}{2} \\ f^{-1}(x) &= \frac{x+7}{2} \end{aligned}$$

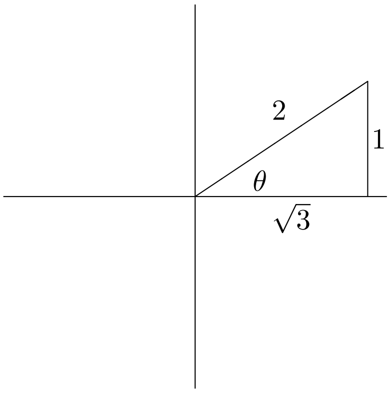
(b) $g(x) = \dfrac{x-1}{x+1}$

$$\begin{aligned} y &= \frac{x-1}{x+1} \\ y(x+1) &= \frac{x-1}{x+1}(x+1) \\ yx+y &= x-1 \\ yx+y-x &= x-1-x \\ yx+y-x &= -1 \\ yx+y-x-y &= -1-y \\ yx-x &= -1-y \\ x(y-1) &= -1-y \\ \frac{x(y-1)}{y-1} &= \frac{-1-y}{y-1} \\ x &= -\frac{1+y}{y-1} \\ f^{-1}(x) &= -\frac{1+x}{x-1} = \frac{1+x}{1-x} \end{aligned}$$

6. Find exact values in radians for the following expressions.

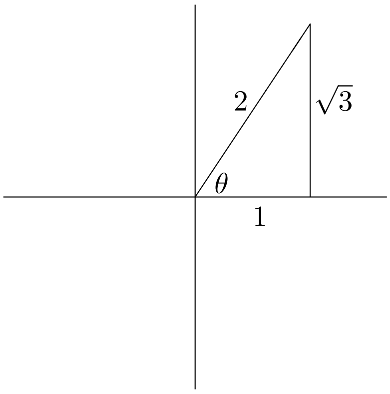
(a) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$\cos^{-1} x$ gives us an angle between 0 and π. Since $\frac{\sqrt{3}}{2}$ is positive, the angle must be in quadrant I. This gives us the picture below.



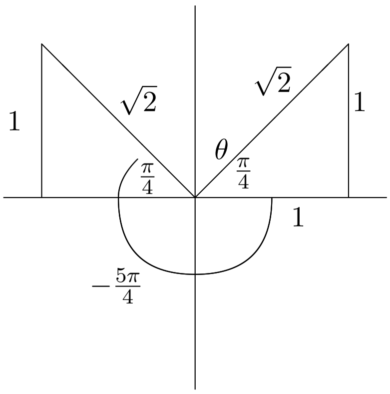
These are the dimensions of the standard 30-60-90 triangle, so $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 30° = \frac{\pi}{6}$.

(b) $\tan^{-1}(\sqrt{3})$
$\tan^{-1} x$ gives us an angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Since $\sqrt{3}$ is positive, the angle must be in quadrant I. This gives us the picture below.



These are the dimensions of the standard $30-60-90$ right triangle, so $\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$.

(c) $\sin^{-1}\left(\sin -\frac{5\pi}{4}\right)$



By definition, $\sin^{-1} x$ must be an angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. We are looking for an angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ which has the same sine as the angle $-\frac{5\pi}{4}$. Since $\sin -\frac{5\pi}{4}$ is positive, the angle we are looking for must be in quadrant I. Since this angle must have the same reference angle as $-\frac{5\pi}{4}$, the angle must be

$$\sin^{-1}\left(\sin -\frac{5\pi}{4}\right) = \frac{\pi}{4}$$

7. Use your calculator to evaluate the following in degrees.

   (a) $\cos^{-1}(.37) \approx$ __68.28°____

   (b) $\tan^{-1}(2.11) \approx$ __64.64°____